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STUDY OF QUEUING BEHAVIOUR UNDER THE ENVIRONMENT OF FUZZY LOGICS

Ravi Siwach

Mail: ravisiwach008@gmail.com

Guide Name: Dr. Vinod Kumar

Abstract

Queuing Behaviour systems are analyzed in banks, hospitals, restaurants, transport systems, service counters, computer systems, and other industrial systems using queueing theory. Most classical queueing models start with the assumption that request/response characteristics are known precisely. In actual applications, however, the above parameters may not be well known, may be ambiguous, or may be expressed in natural language. The uncertainty can be represented by fuzzy numbers and fuzzy function in fuzzy logic representation. This paper studies the behaviour of M/M/1and M/M/2queueing models under a fuzzy environment. The arrival rate and service rate are represented as triangular fuzzy numbers. The fuzzy performance measures obtained are traffic intensity, number of customers in queue, number of customers (Q) in system, waiting time (w) in queue and waiting time (W) in system using the α-cut method and the DSW algorithm. Finally, a numerical example is used for comparison of fuzzy M/M/1 and fuzzy M/M/2 models. As the number of servers is increased, the size of the queues decreases as does the waiting time, as indicated by the results. So fuzzy queueing models are a more valid technique for analysing systems with an uncertain behaviour of arrival and service conditions.

Keywords: fuzzy logic, queueing behaviour, $M/M/1$, $M/M/2$, triangular fuzzy number, α-cut, DSW algorithm, waiting time, queue length.

1. Introduction

A mathematical tool employed in the study of Queuing Behaviour systems is called queueing theory. Customers come in need in a queueing system to a way service may be busy and customers have to wait. These systems are available in most restaurants, banks, hospitals, toll plazas, railway stations, call centers, computer systems and manufacturing units. In many of the traditional queueing models the interarrival times are assumed to be independently and identically distributed by a Poisson process and service times are assumed be. in the exponential distribution. The assumptions help the mathematician solve the problem mathematically, but may not be an accurate representation of a real-life situation. The rate of arrivals and the service rate in many practical situations may not be known precisely. These can change because of human behavior, workload, service speed, traffic, machine condition or because customers tend to change their requirements. For this purpose, fuzzy logic can be helpful as it mathematically represents the uncertainty. The fuzzy numbers are used to represent the single valued arrival rate or service rate. For a fuzzy number, there is a range of values and a level of belonging (also called membership grade). In this way, fuzzy queueing model is more flexible and realistic than crisp queueing model.

The concept of fuzzy sets was introduced by Zadeh[1]. Later, possibility theory was developed as an extension of fuzzy set theory[2]. Several researchers have applied fuzzy logic to queueing models. Buckley studied elementary queueing theory using possibility theory[3]. Prade discussed fuzzy and possibilistic models for queueing systems[4]. Li and Lee analyzed fuzzy queues using the extension principle[5]. Negi and Lee used simulation and α-cut methods to analyze fuzzy queues[6]. Kao, Li, and Chen used parametric programming to study fuzzy queueing systems[7]. Chen also studied fuzzy queues with finite capacity and bulk service[8].

This paper focuses on the fuzzy behaviour of $M/M/1$and $M/M/2$queueing models. The $M/M/1$model contains one server, while the $M/M/2$model contains two servers. Both models are analyzed using triangular fuzzy arrival and service rates. The performance measures are calculated at different α-cut levels and then compared.

2. Basic Queueing Models

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[1] Zadeh LA. Fuzzy sets. Information and Control. 1965;8(3):338-353.
[2] Zadeh LA. Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems. 1978;1(1):3-28.
[3] Buckley JJ. Elementary queueing theory based on possibility theory. Fuzzy Sets and Systems. 1990;37(1):43-52.
[4] Prade HM. An outline of fuzzy or possibilistic models for queueing systems. In: Wang PP, Chang SK, editors. Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems. New York: Plenum Press; 1980. p.147-153.
[5] Li RJ, Lee ES. Analysis of fuzzy queues. Computers & Mathematics with Applications. 1989;17(7):1143-1147.
[6] Negi DS, Lee ES. Analysis and simulation of fuzzy queues. Fuzzy Sets and Systems. 1992;46(3):321-330.
[7] Kao C, Li CC, Chen SP. Parametric programming to the analysis of fuzzy queues. Fuzzy Sets and Systems. 1999;107(1):93-100.
[8] Chen SP. Parametric nonlinear programming approach to fuzzy queues with bulk service. European Journal of Operational Research. 2005;163(2):434-444.

2.1 Meaning of $M/M/1$ Model

The $M/M/1$ model is a single-server queueing model. It assumes that customers arrive according to a Poisson process and service times follow an exponential distribution.[9]

The notation $M/M/1$ means:

$$M = \text{Markovian arrival process}$$

$$M = \text{Markovian service process}$$

$$1 = \text{one server}$$

The model is commonly used for systems such as a single service counter, a small restaurant counter, a repair desk, or a single-machine service system.

The main assumptions of the $M/M/1$ model are:

1. Inter-arrival times are exponentially distributed.
2. Service times are exponentially distributed.
3. Customers are served according to the first-come, first-served discipline.
4. There is only one server.
5. The queue capacity is infinite.
6. The arrival rate must be less than the service rate for the system to remain stable.

2.2 Meaning of $M/M/2$ Model The $M/M/2$ model is a two-server queueing model. It assumes that arrivals follow a Poisson process and service times are exponentially distributed. The system contains two identical servers.[10]

[9] Dubois D, Prade H. Fuzzy Sets and Systems: Theory and Applications. New York: Academic Press; 1980.
[10] Kaufmann A. Introduction to the Theory of Fuzzy Subsets. New York: Academic Press; 1975.

The notation $M/M/2$means:

$$M = \text{Markovian arrival process}$$

$$M = \text{Markovian service process}$$

$$2 = \text{two servers}$$

The $M/M/2$model is useful for service systems where two counters or two service channels are available. Examples include restaurants with two billing counters, banks with two service desks, hospitals with two registration counters, or computer systems with two processors.

The stability condition for the $M/M/2$model is:

$$\rho = \frac{\lambda}{2\mu} < 1$$

where $\lambda$is the arrival rate and $\mu$is the service rate of each server.

3. Fuzzy Logic and Queueing Behaviour

In a classical queueing model, arrival rate and service rate are treated as exact values. For example:

$$\lambda = 3, \mu = 7$$

In a fuzzy queueing model, these values are represented by fuzzy numbers. For example:

$$\tilde{\lambda} = (2,3,4)$$

$$\tilde{\mu} = (6,7,8)$$

This means that the arrival rate is most likely 3, but it may vary from 2to 4. Similarly, the service rate is most likely 7, but it may vary from 6to 8.

This approach is useful because in real systems, arrival and service rates are rarely fixed. They may change due to time, customer load, working speed, machine condition, or external factors.[11]

4. Triangular Fuzzy Number

A triangular fuzzy number is represented by three real numbers:

$$\widetilde{U} = (u_1, u_2, u_3)$$

where:

$$u_1 = \text{lower value}$$

$$u_2 = \text{most likely value}$$

$$u_3 = \text{upper value}$$

The membership function of a triangular fuzzy number is:

$$\mu_{\widetilde{U}}(x) = \begin{cases} 0, & x \le u_1 \\ \dfrac{x - u_1}{u_2 - u_1}, & u_1 \le x \le u_2 \\ 1, & x = u_2 \\ \dfrac{x - u_3}{u_2 - u_3}, & u_2 \le x \le u_3 \\ 0, & \text{otherwise} \end{cases}$$

The triangular fuzzy number increases from 0to 1, reaches its highest membership value at u_2, and then decreases from 1to 0.

5. α-Cut Method

The α-cut method converts a fuzzy number into an interval. If:

$$\widetilde{U} = (u_1, u_2, u_3)$$

[11] Kaufmann A, Gupta MM. Introduction to Fuzzy Arithmetic: Theory and Applications. New York: Van Nostrand Reinhold; 1985.

then the α-cut is:

$$U^{\alpha} = [u_1 + \alpha(u_2 - u_1),\ u_3 - \alpha(u_3 - u_2)]$$

where:

$$0 \le \alpha \le 1$$

When $\alpha = 0$, the widest interval is obtained. When $\alpha = 1$, the most certain value is obtained.

For the triangular fuzzy arrival rate:

$$\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$$

the α-cut is:

$$\lambda^{\alpha} = [\lambda_1 + \alpha(\lambda_2 - \lambda_1),\ \lambda_3 - \alpha(\lambda_3 - \lambda_2)]$$

For the triangular fuzzy service rate:

$$\tilde{\mu} = (\mu_1, \mu_2, \mu_3)$$

the α-cut is:

$$\mu^{\alpha} = [\mu_1 + \alpha(\mu_2 - \mu_1),\ \mu_3 - \alpha(\mu_3 - \mu_2)]$$

6. Fuzzy Arithmetic

Let:

$$\tilde{U} = (u_1, u_2, u_3)$$

And

$$\tilde{V} = (v_1, v_2, v_3)$$

be two triangular fuzzy numbers.

6.1 Addition

$$\widetilde{U} + \widetilde{V} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$$

6.2 Negative of a Fuzzy Number

$$-\widetilde{V} = (-v_3, -v_2, -v_1)$$

6.3 Subtraction

$$\widetilde{U} - \widetilde{V} = (u_1 - v_3,\ u_2 - v_2,\ u_3 - v_1)$$

6.4 Multiplication

For positive fuzzy numbers:

$$\widetilde{U} \times \widetilde{V} = (u_1 v_1,\ u_2 v_2,\ u_3 v_3)$$

6.5 Reciprocal

If $v_1, v_2, v_3$are non-zero positive real numbers, then:

$$\frac{1}{\widetilde{V}} = \left(\frac{1}{v_3}, \frac{1}{v_2}, \frac{1}{v_1}\right)$$

6.6 Division

$$\frac{\widetilde{U}}{\widetilde{V}} = \left(\frac{u_1}{v_3}, \frac{u_2}{v_2}, \frac{u_3}{v_1}\right)$$

6.7 Scalar Multiplication

For a real number K:

$$K\widetilde{U} = (Ku_1, Ku_2, Ku_3), K \geq 0$$

$$K\widetilde{U} = (Ku_3, Ku_2, Ku_1), K < 0$$

7. DSW Algorithm

The DSW algorithm is used to calculate membership functions of fuzzy performance measures. It uses α-cut intervals and interval arithmetic.[12]

The steps are:

Step 1: Select the value of αin the range:

$$[0, 1]$$

Step 2: Locate the input membership-function intervals corresponding to the selected α-cut.

Step 3: Use interval arithmetic to calculate the output interval for that α-cut.

Step 4: Repeat the process for different values of $\alpha$to obtain the complete fuzzy solution.

8. Fuzzy $M/M/1$Queueing Model

For the classical $M/M/1$model, the traffic intensity is:

$$\rho = \frac{\lambda}{\mu}$$

The system is stable if:

$$\rho < 1$$

The average number of customers in the system is:

$$L_s = \frac{\lambda}{\mu - \lambda}$$

[12] Zimmermann HJ. Fuzzy Set Theory and Its Applications. Dordrecht: Kluwer Academic Publishers; 1991.

The average number of customers in the queue is:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

The average waiting time in the system is:

$$W_s = \frac{1}{\mu - \lambda}$$

The average waiting time in the queue is:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

In the fuzzy $M/M/1$model, $\lambda$and $\mu$are fuzzy numbers. Therefore, the performance measures also become fuzzy.

9. Numerical Example for Fuzzy $M/M/1$Model

Let the fuzzy arrival rate and fuzzy service rate be:

$$\tilde{\lambda} = (2,3,4)$$

$$\tilde{\mu} = (6,7,8)$$

The α-cut of arrival rate is:

$$\lambda^\alpha = [2 + \alpha,\ 4 - \alpha]$$

The α-cut of service rate is:

$$\mu^\alpha = [6 + \alpha,\ 8 - \alpha]$$

Using these values, the performance measures of the fuzzy $M/M/1$model are calculated.[13]

Table 1. Performance Measures of Fuzzy $M/M/1$Model

α	λ^{α}	μ^{α}	L_s	L_q	W_s	W_q
0	[2.0, 4.0]	[6.0, 8.0]	[0.33, 2.00]	[0.083, 1.33]	[0.082, 1.00]	[0.0207, 0.665]
0.2	[2.2, 3.8]	[6.2, 7.8]	[0.39, 1.58]	[0.110, 0.967]	[0.100, 0.720]	[0.028, 0.440]
0.4	[2.4, 3.6]	[6.4, 7.6]	[0.46, 1.28]	[0.145, 0.723]	[0.130, 0.530]	[0.040, 0.301]
0.8	[2.8, 3.2]	[6.8, 7.2]	[0.63, 0.88]	[0.247, 0.418]	[0.190, 0.310]	[0.059, 0.149]
1	[3.0, 3.0]	[7.0, 7.0]	[0.75, 0.75]	[0.321, 0.321]	[0.250, 0.250]	[0.107, 0.107]

From Table 1, the most expected value of $L_s$at $\alpha = 1$is:

$$[0.75, 0.75]$$

and it is impossible for the value to fall outside:

$$[0.33, 2.00]$$

Similarly, the maximum value of $L_q$is:

$$0.321$$

and it is impossible for it to fall outside:

[13] Dong WM, Shah HC, Wong FS. Fuzzy computations in risk and decision analysis. Civil Engineering Systems. 1985;2(4):201-208.

[0.083, 1.33]

The waiting time in the system and waiting time in the queue are also reduced as uncertainty decreases.[14]

10. Fuzzy $M/M/2$ Queueing Model

The $M/M/2$ model contains two servers. For this model, the traffic intensity is:

$$\rho = \frac{\lambda}{2\mu}$$

The probability of zero customers in the system is:

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2}{2!} \cdot \frac{2\mu}{2\mu - \lambda}}$$

The average number of customers in the queue is:

$$L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^2}{(2-1)!\,(2\mu - \lambda)^2} P_0$$

The average number of customers in the system is:

$$L_s = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^2}{(2-1)!\,(2\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu}$$

The average waiting time in the queue is:

$$W_q = \frac{\mu\left(\frac{\lambda}{\mu}\right)^2}{(2-1)!\,(2\mu - \lambda)^2} P_0$$

[14] Buckley JJ, Feuring T, Hayashi Y. Fuzzy queueing theory revisited. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 2001;9(5):527-537.

The average waiting time in the system is:

$$W_s = \frac{\mu\left(\frac{\lambda}{\mu}\right)^2}{(2-1)!\,(2\mu-\lambda)^2} P_0 + \frac{1}{\mu}$$

In fuzzy form, $\lambda$and $\mu$are replaced by $\lambda^\alpha$and μ^α.

11. Numerical Example for Fuzzy $M/M/2$Model"

The same fuzzy arrival rate and service rate are considered:

$$\tilde{\lambda} = (2,3,4)$$
$$\tilde{\mu} = (6,7,8)$$

The α-cut forms are:

$$\lambda^\alpha = [2+\alpha,\ 4-\alpha]$$
$$\mu^\alpha = [6+\alpha,\ 8-\alpha]$$

The calculated fuzzy performance measures for the $M/M/2$model are shown in Table 2.

Table 2. Performance Measures of Fuzzy $M/M/2$Model

α	λ^α	μ^α	L_q	L_s	W_s	W_q
0	[2.0, 4.0]	[6.0, 8.0]	[0.002, 0.171]	[0.252, 0.831]	[0.125, 0.251]	[0.0001, 0.085]
0.2	[2.2, 3.8]	[6.2, 7.8]	[0.003, 0.112]	[0.285, 0.724]	[0.129, 0.211]	[0.0007, 0.050]
0.4	[2.4, 3.6]	[6.4, 7.6]	[0.004, 0.071]	[0.319, 0.632]	[0.132, 0.185]	[0.0011, 0.029]
0.8	[2.8, 3.2]	[6.8, 7.2]	[0.012, 0.030]	[0.401, 0.501]	[0.142, 0.157]	[0.0037, 0.010]

1	[3.0, 3.0]	[7.0, 7.0]	[0.020, 0.020]	[0.448, 0.448]	[0.148, 0.148]	[0.006, 0.006]

In Table 2, the performance measures of the fuzzy M/M/2 queueing model are shown at various levels of α-cut. According to this model, the arrival rate λ^αand the service rate μ^αare taken as intervals as a consequence of the uncertainty is considered by means of fuzzy logic. The fuzzy intervals become narrower and narrower as the value of α increases from 0 to 1, thus bringing down uncertainty gradually. If α = 0, then outputs a much broader range of values, and if α = 1, then the output range is the most expected and most precise one. Performance Measures are: Average Number of Customers in the queue L_q, Average Number of Customers in the system L_s, Average Waiting time in the system W_s, Average Waiting time in the queuing W_q From the table, it is seen that the most expected value of L^s at α = 1 is [0.448, 0.448] and it cannot be changed to values more than [0.252, 0.831]. Similarly, the most expected value of L_q is [0.020, 0.020], and its possible range is limited between [0.002, 0.171]. The most expected values of W_s and W_q are [0.148, 0.148] and [0.006, 0.006], respectively. The above results demonstrate the efficiency of this fuzzy M/M/2 system as the queue length and system's waiting time are very low. By having two servers, customers can be served quicker which makes the system less congested.

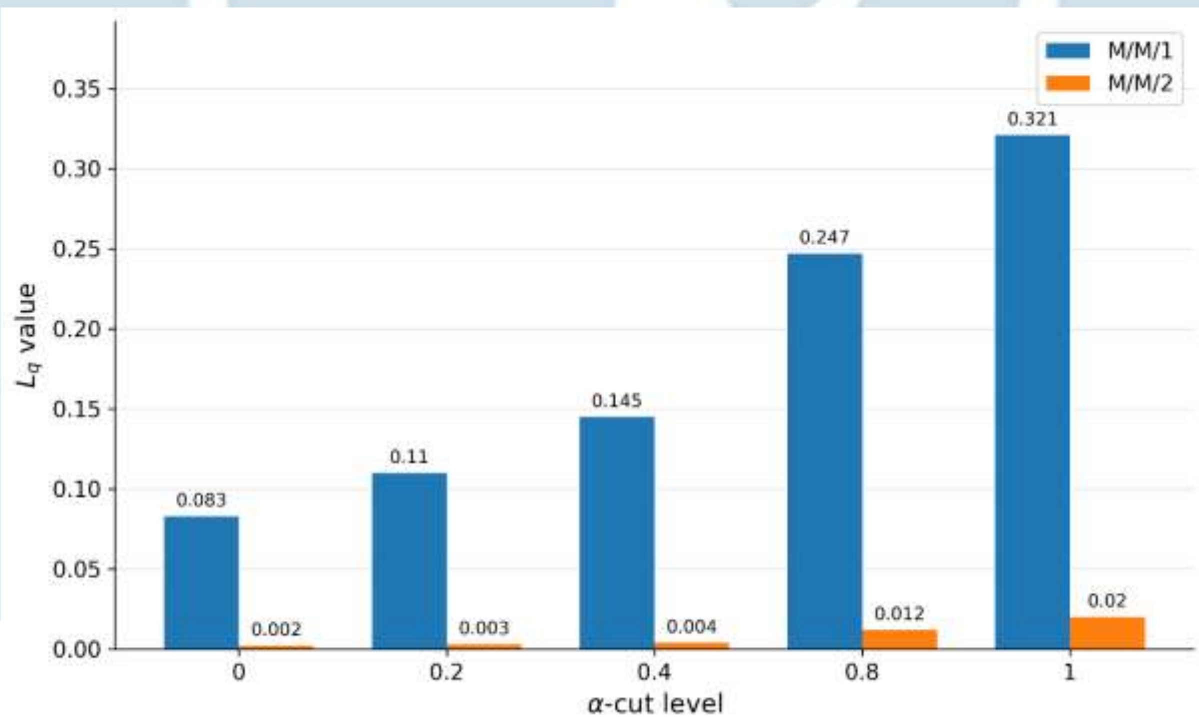

Figure 1. Minimum Value of L_q

The comparison between the minimum value of the average number of customers in the queue (L_q) of the queueing models (M/M/1) and (M/M/2) is shown in figure 1. It is clearly observed from the graph that for all α -cut values the minimum value of L_q is less in the fuzzy M/M/2 model than its value in the fuzzy M/M/1 model. It implies that in the case of 2 servers fewer customers will stand waiting in the line. The fuzzy M/M/1 model has just one server serving the arriving customers and consequently, with a higher arrival rate, customers might have to wait for a longer period in the model. But in the fuzzy M/M/2 model, service facility is improved due to the fact that two servers are working in unison. This benefits the customer because they are served faster and the line does not grow to its extent. This number, then, shows that the two servers allow the shortest queue length and that the system performs better when fuzzy.

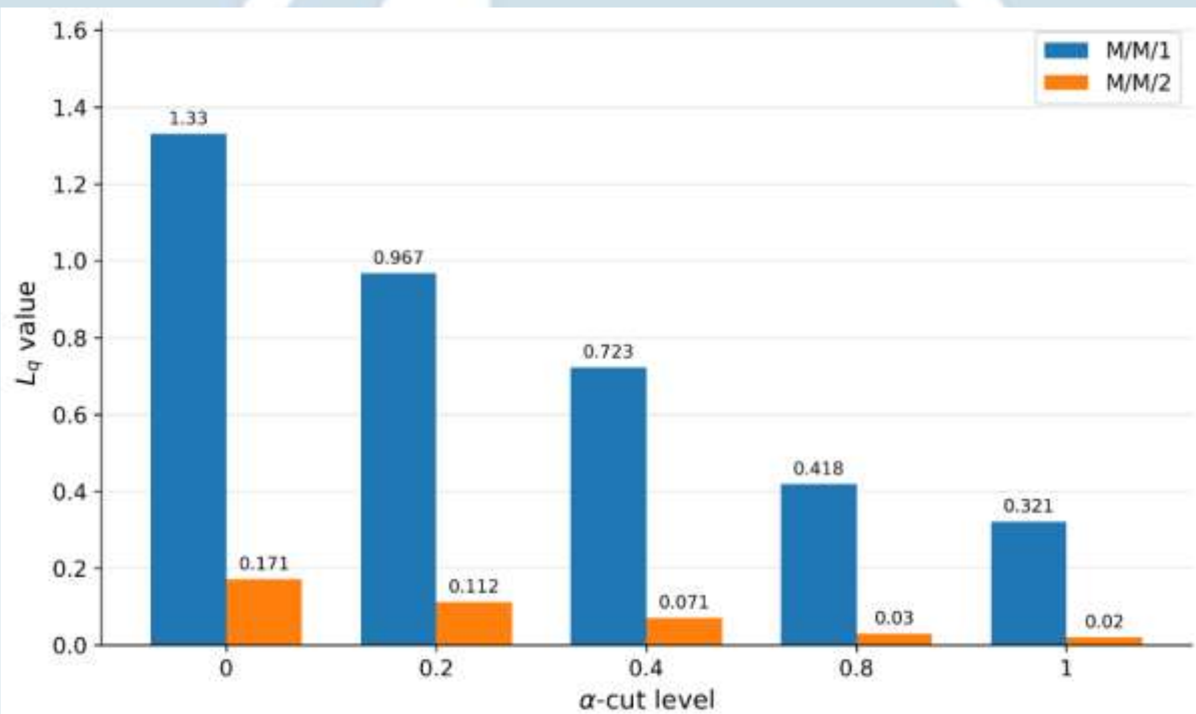

Figure 2. Maximum Value of L_q

The maximum of L_q for of the fuzzy M/M/1 and fuzzy M/M/2 models is compared in figure 2. The greatest value is the maximum average number of waiters in the queue in spite of the uncertainty. It is easy to see from the figure that the max value of L_q of the fuzzy M/M/1 model is higher than the fuzzy M/M/2 model at each α-cut value. This indicates that if one were to take into account uncertainty on average expected arrival and service rates, a single-server configuration would be expected to have longer queues. The fuzzy M/M/2 model, on the other hand, yields a lower maximum values since the second server helps to split the load from customer. While one server is busy serving the next customer, the second server may serve the next customer, thereby not allowing queueing to build up and reducing waiting times. Hence, this number strongly advocates the two-server model to be more suitable for the queue congestion reduction adaptable.

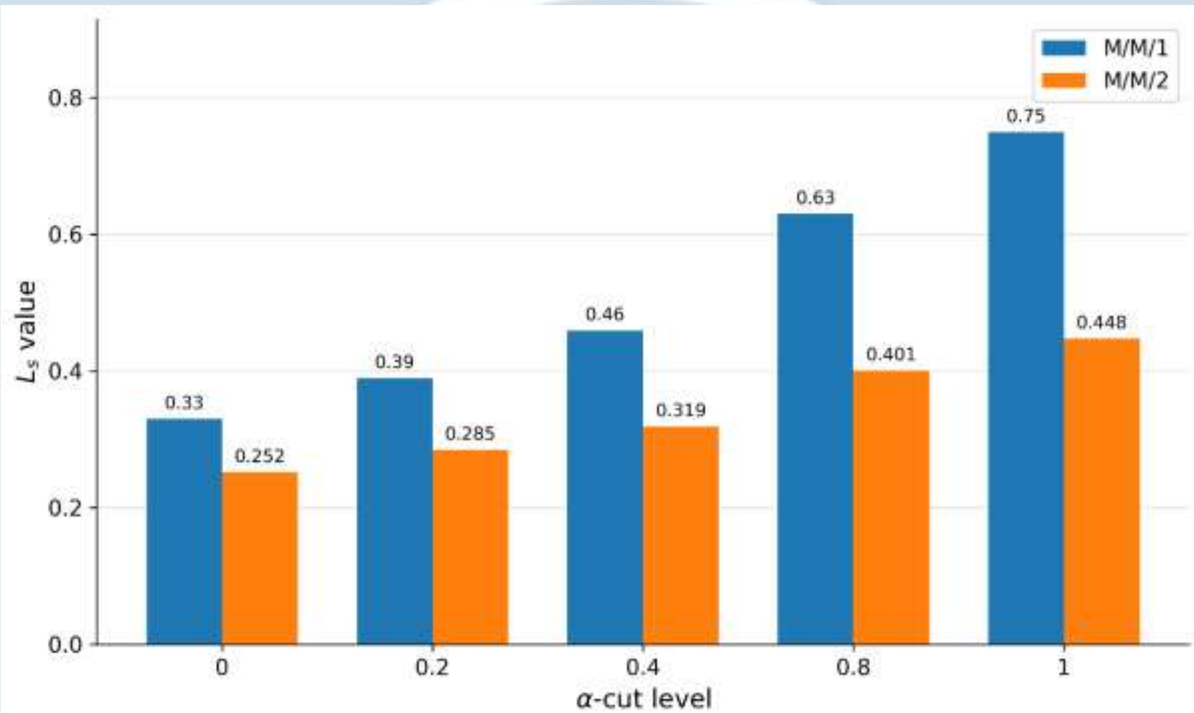

Figure 3. Minimum Value of L_s

The general comparison of average number of customers in system (L_s) in case of fuzzy M/M/1 and fuzzy M/M/2 models is made by comparing their minimum values as displayed in figure 3. The value of L^s consists of customers waiting in the queue, and the customers being served. As is shown in figure, the minimum value of L_{st} for different α-cuts is higher for fuzzy M/M/1 model than for fuzzy M/M/2 model. Thus in a single-server system, more customers stay in the system since the service process is slower. The model M/M/2 on the other hand works and makes fewer customers present in the system since two servers can serve more people arriving than in the preceding model. The resulting sum of the two-server system remains the same as α increases, but the values become more exact the greater the value of α. Based on this result, the number of server reduces to half when they increases from one to two, which means that the average number of customers in the system decreases.

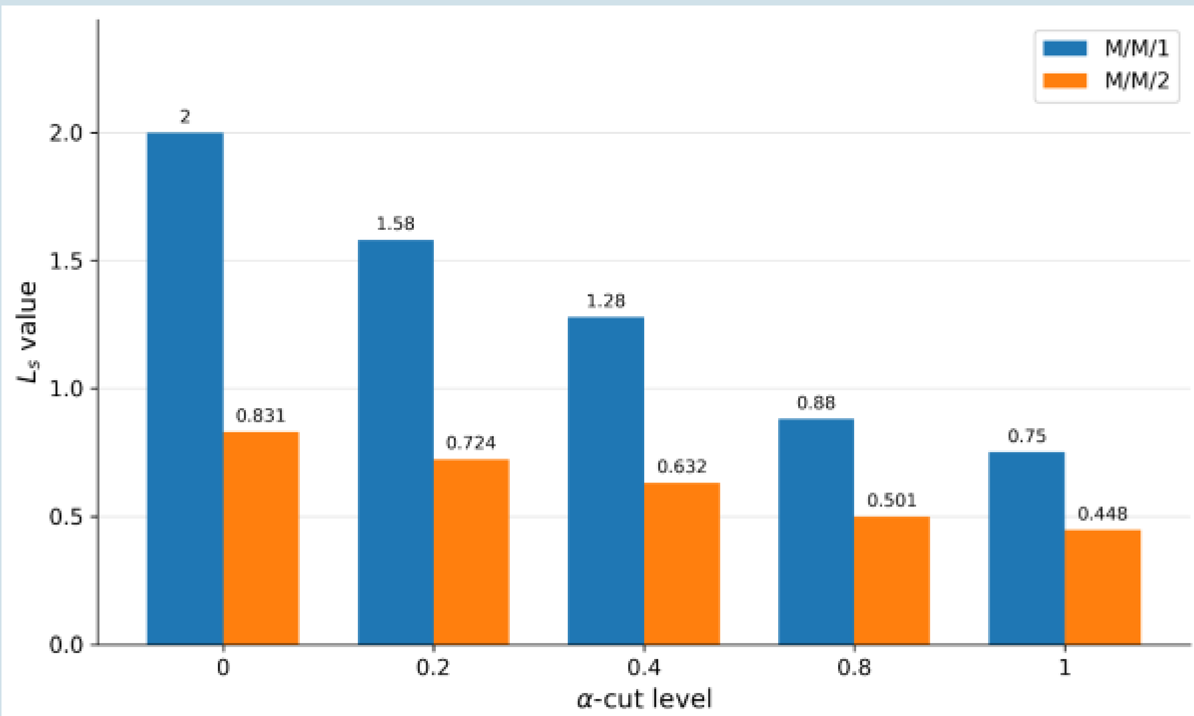

Figure 4. Maximum Value of L_s

In Fig. 4 the Maximum L_s for of the fuzzy M/M/1 queueing model is compared to the fuzzy M/M/2 queueing model. The max of L_s is important as in fuzzy uncertainty, it presents the maximum number of customers that can exist within the system. The figure shows that the maximum values of L_s are larger in the fuzzy M/M/1 model, this means that the number of customers in the system is greater when the model only has one server. This is because customers need to wait longer for service, particularly with high arrival rates or when the service rate is low. The maximum values are reduced in the fuzzy M/M/2 model since it involves having two servers available which does increase the service capacity. The second server contributes to minimizing total time customers are spending in the system. Hence it is being proved that the fuzzy M/M/2 system is better in controlling congestion in the system and effective in improving customer flow.

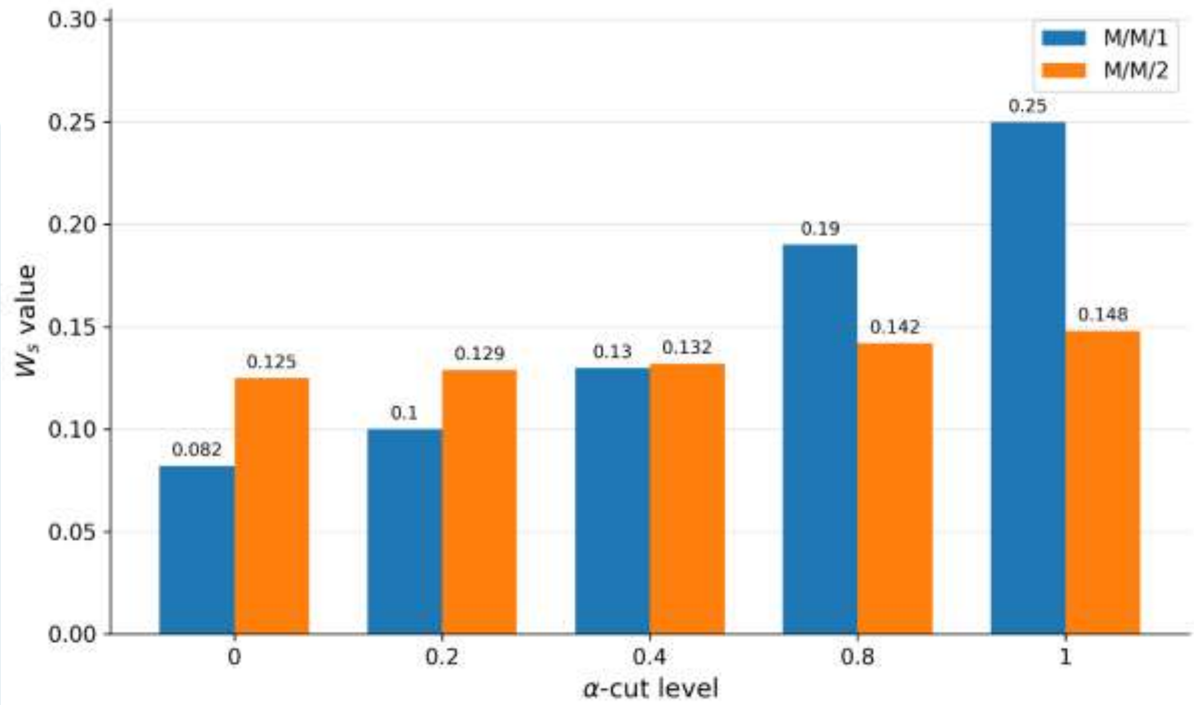

Figure 5. Minimum Value of W_s

The minimum values of the average waiting time in the system (W_s) for the fuzzy M/M/1 and fuzzy M/M/2 models are compared in figure 5. Waiting time in the system is the sum of waiting time in queue and service time. From the figure, it is seen that the performance of waiting-time for fuzzy M/M/2 is better at higher α-cut values. In some instances, the minimum value of W_s for the M/M/2 model can be slightly more than the uncertainty reduces and the α-cut value increases. The results show that the fuzzy M/M/2 system offers higher service reliability as the values become more regular and accurate. With two servers, the customers can be served quicker and their total time in the system is decreased. Hence, it is evident that the fuzzy M/M/2 model helps to enhance the service process and delay time of customers.

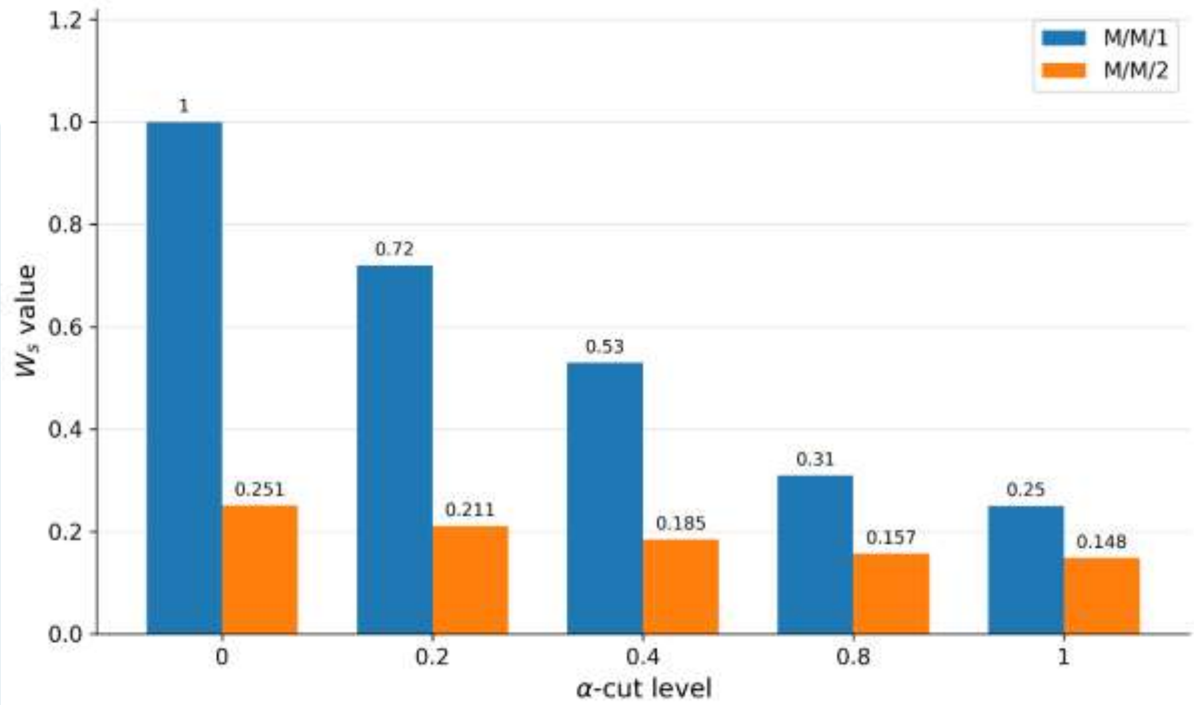

Figure 6. Maximum Value of W_s

The comparison between the values of the W_s at maximum for the fuzzy M/M/1 and fuzzy M/M/2 is plotted in figure 6. W_s represents the maximum waiting time that a customer can experience in the system. The figure shows that the fuzzy M/M/1 model has much higher maximum values of W_s than the fuzzy M/M/2 model. This implies that customers may spend more time in the system if there is only one server. In a single-server system, when the server is already in service, the arriving customers will have to wait until the service has finished. This adds up the waiting time. But in the fuzzy M/M/2 model, this problem is decreased by the availability of two servers. There are two servers, with which customers can be served, which reduces the maximum time spend in the system. Therefore, this number supports that the two-server system performs more efficiently in minimizing the overall waiting time of the customers.

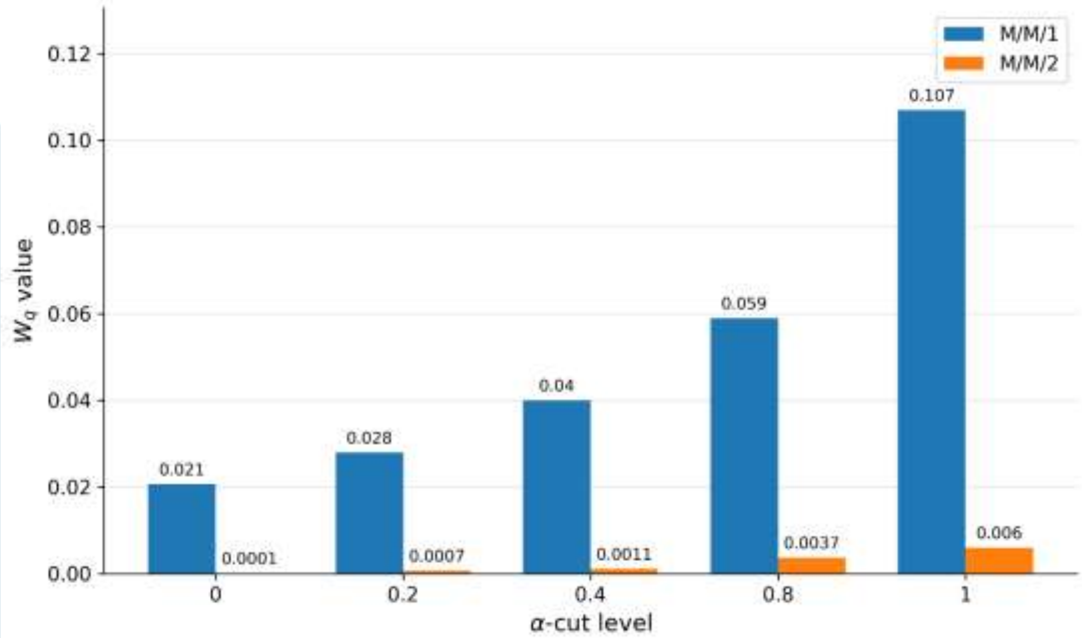

Figure 7. Minimum Value of W_q

The comparison of minimum value of average waiting time in queue (W_q) for fuzzy M/M/1 queueing model and fuzzy M/M/2 queueing model is given in figure 7. In the value of W_q only, the time spent waiting before service starts is included. The figure shows that the value of W_q for the fuzzy M/M/2 model is less than that of fuzzy M/M/1 model. This means that if there are two servers, the customer has to wait for a shorter time in the queue. The second server alleviates the load on the first server and enables customers to be in service faster. In the fuzzy M/M/2 model, consequently, the waiting time in the queue is very small. The figure clearly shows that the waiting time for service in the two-server system is lower compared to the single-server system.

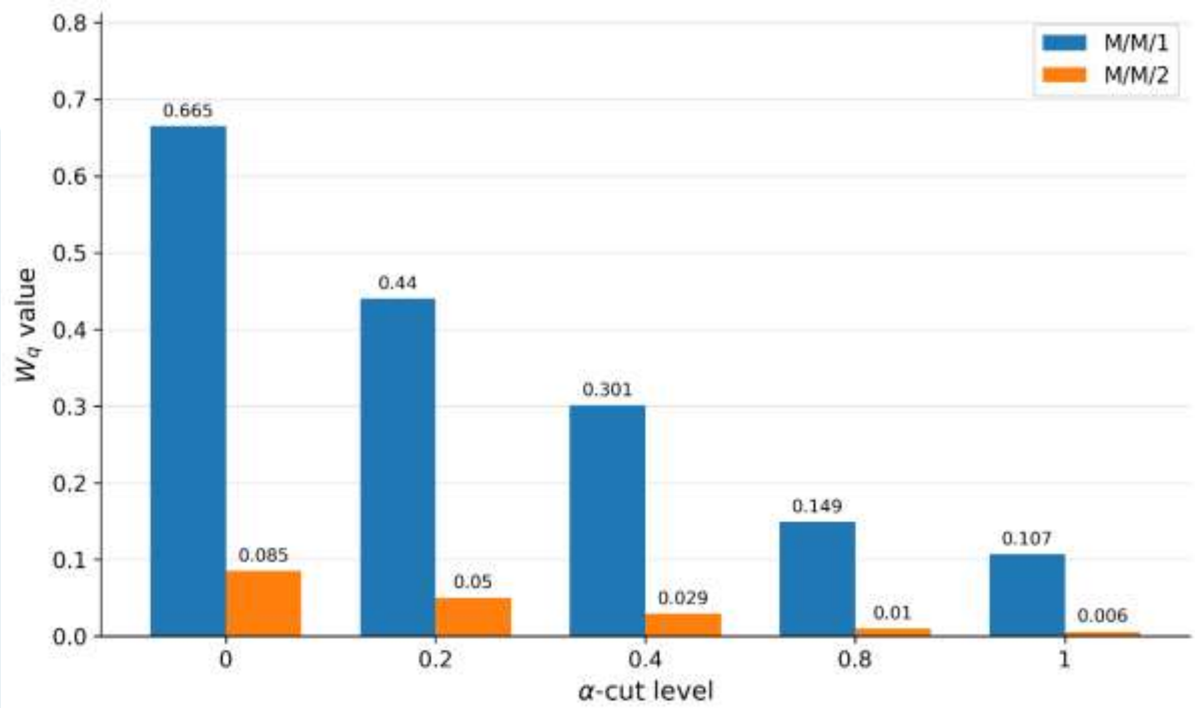

Figure 8. Maximum Value of W_q

The maximum values of W_q for fuzzy M/M/1 model and fuzzy M/M/2 model are compared with each other in figure 8. W_q shows the maximum value of the waiting time in the queue when it is subjected to fuzzy uncertainty. As indicated in the figure, the maximum performance (W_q is) of the model M/M/1 is greater than that of the model M/M/2 in the fuzzy model. This implies that customers will have to wait longer in the single server system. If only one server is available, the customer has to wait if there's already a customer on the server, thus resulting in a longer queue delay. In the fuzzy M/M/2 model, there are two servers, which means that customers are less likely to wait for a long time. The second server has the effect of increasing the capacity of the service and shortening the service queue delay. Hence, this number verifies that one more server leads to a decrease in the waiting time at the queue and hence results in an improvement in the overall performance of the queueing system.

12. Comparison of Fuzzy $M/M/1$and Fuzzy $M/M/2$Models

The comparison between fuzzy $M/M/1$and fuzzy $M/M/2$models is made using the lower and upper values of the main performance measures, namely L_q, L_s, W_s, and W_q. Here, $L_q$represents the average number of customers in the queue, $L_s$represents the average number of customers in the system, $W_s$represents the average waiting time in the system, and $W_q$represents the average waiting time in the queue.[15]

Table 3. Comparison of Minimum and Maximum Values of L_q

α	$M/M/1$**Minimum** L_q	$M/M/2$**Minimum** L_q	$M/M/1$**Maximum** L_q	$M/M/2$**Maximum** L_q
0	0.083	0.002	1.330	0.171
0.2	0.110	0.003	0.967	0.112
0.4	0.145	0.004	0.723	0.070
0.8	0.247	0.012	0.418	0.030
1	0.321	0.020	0.321	0.020

The minimum and maximum values of L_q for the fuzzy M/M/1 and fuzzy M/M/2 models at various α-cut levels are presented in Table 3. The table clearly shows that both the minimum and maximum values of L_q are lower in the fuzzy M/M/2 model than in the fuzzy M/M/1 model. The two-server model will give a reduction in the expected number of customers in the queue. At $\alpha = 0$, the maximum value of L_q for the M/M/1 model is 1.330, while for the M/M/2 model it is only 0.171. This is a significant drop that demonstrates significant impact of adding a second server. At $\alpha = 1$, the most expected value of L_q is 0.321 for the M/M/1 model and only 0.020 for the M/M/2 model. It is just a confirmation of the fact that the fuzzy M/M/2 model provides a better performance of the queues. Overall the table shows that the two-server configuration is more effective in reducing the queue length and controlling congestion.

[15] Gross D, Shortle JF, Thompson JM, Harris CM. Fundamentals of Queueing Theory. 4th ed. Hoboken: John Wiley & Sons; 2008.

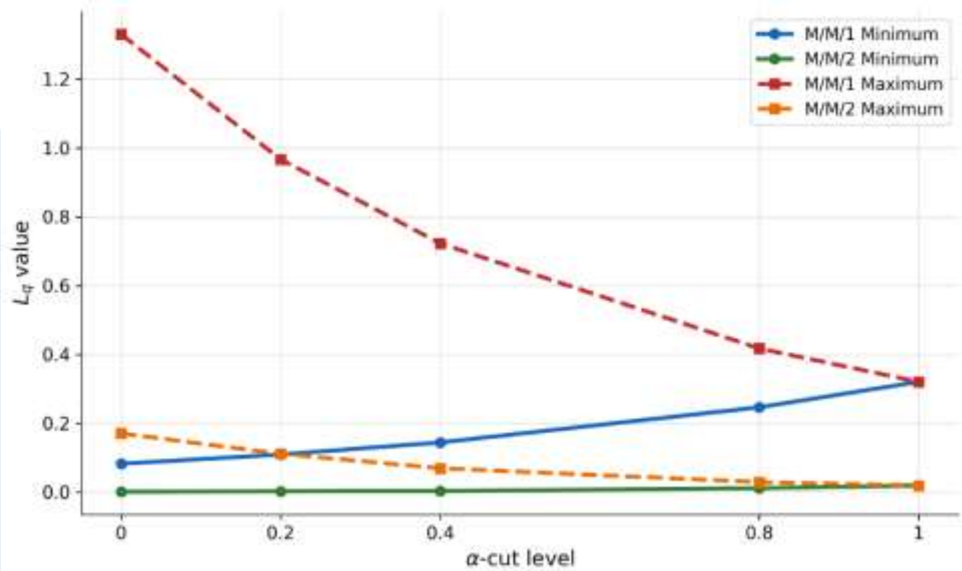

Figure 9. Comparison of Minimum and Maximum Values of L_q for Fuzzy $M/M/1$ and Fuzzy $M/M/2$ Models

The comparison between the lower and upper limits for L_q for the fuzzy M/M/1 queueing model and fuzzy M/M/2 queueing model is presented. As seen in the figure, the values of the queue length for the fuzzy M/M/2 model is smaller than those of the fuzzy M/M/1 model in both minimum and the maximum case. This indicates that the two-server system is better in terms of the number of customers that will be waiting in the queue. The difference between the models M/M/1 and M/M/2 is more pronounced in the maximum values, the latter with much higher queue congestion. The M/M/2 model has two service channels and can be able to serve more customers in less time. This means the queue is now shorter and everything works better. This number reinforces the observation that there is much to gain in terms of queueing performance from adding another server.

Table 4. Comparison of Minimum and Maximum Values of L_s

α	$M/M/1$ **Minimum** L_s	$M/M/2$ **Minimum** L_s	$M/M/1$ **Maximum** L_s	$M/M/2$ **Maximum** L_s
0	0.330	0.252	2.000	0.831

0.2	0.390	0.285	1.500	0.724
0.4	0.460	0.319	1.280	0.632
0.8	0.630	0.401	0.880	0.501
1	0.750	0.448	0.750	0.448

Table 4 gives the minimum and maximum of L_s for the fuzzy M/M/1 and fuzzy M/M/2 models. L_s is the average number of customers in the system, that is, in line and being served. It is observed from the table that the value of L_s is much lower when compared to the fuzzy M/M/1 model in the fuzzy M/M/2 model. At $\alpha = 0$, the maximum value of L_s is 2.000 for the M/M/1 model, while it is only 0.831 for the M/M/2 model. At $\alpha = 1$, the value of L_s is 0.750 for M/M/1 and 0.448 for M/M/2. From these results, one can conclude that once more server is added to the system, the number of customers in the system decreases. The two-server system increases the throughput of the system by minimizing waiting and service congestion. Hence the fuzzy M/M/2 model is more appropriate to obtain the better system performance.

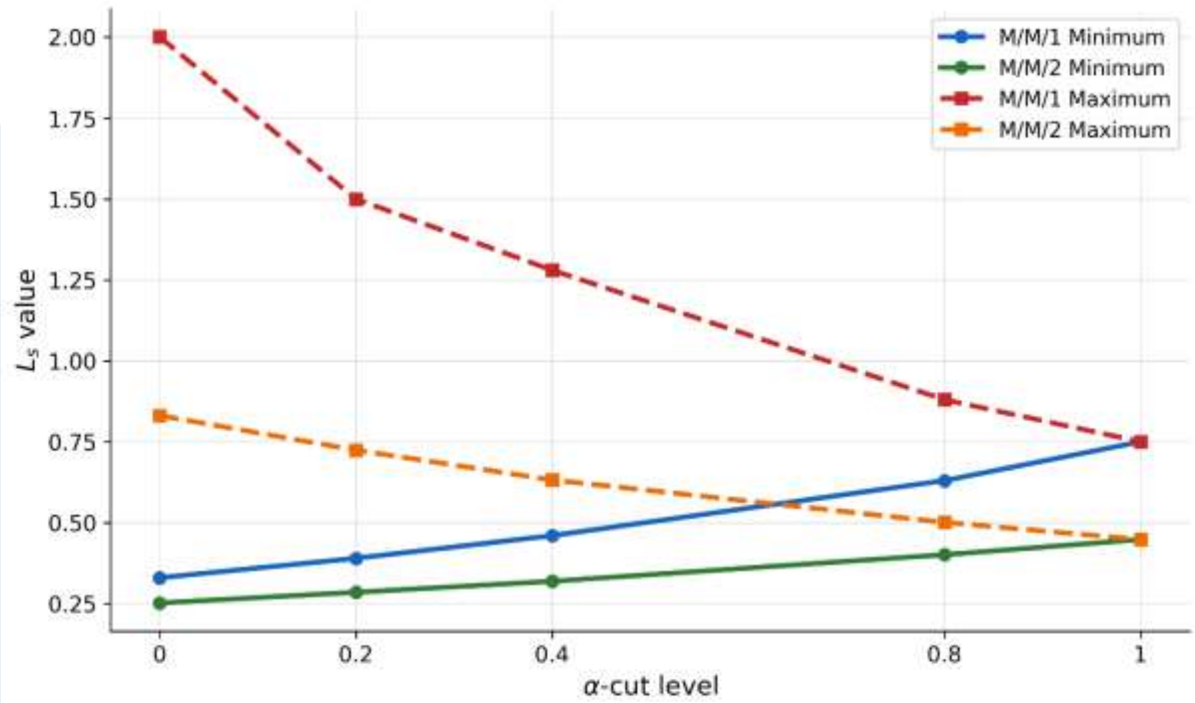

Figure 10. Comparison of Minimum and Maximum Values of $L_s$for Fuzzy $M/M/1$and Fuzzy $M/M/2$Models

This figure shows the minimum and maximum values of L_s for the fuzzy M/M/1 and fuzzy M/M/2. The values for the M/M/2 model are always less than those of the M/M/1 model, as illustrated in the graph. This translates to reduced number of customers in the system when two servers are available. The single server in the M/M/1 model is more likely to be busy, thus triggering longer waiting and/or service times for customers. The introduction of two servers not only increases the service rate, but also decreases the number of customers in the system in the M/M/2 model. The figure indicates, therefore, that the two-server system is more effective in managing the congestion of the system. It also gives a clue that the more servers there are the more efficient the queueing system will be.

Table 5. Comparison of Minimum and Maximum Values of W_s

α	$M/M/1$**Minimum** W_s	$M/M/2$**Minimum** W_s	$M/M/1$**Maximum** W_s	$M/M/2$**Maximum** W_s

0	0.082	0.125	1.000	0.251
0.2	0.100	0.129	0.720	0.211
0.4	0.130	0.132	0.530	0.185
0.8	0.190	0.142	0.310	0.157
1	0.250	0.148	0.250	0.148

The minimum and maximum values of W_s for the fuzzy M/M/1 and fuzzy M/M/2 models are shown in Table 5. W_s is the mean of the waiting time in the system. It is seen that the maximum waiting time in the system is very small in the fuzzy M/M/2 model as compared to the fuzzy M/M/1 model from the table. At $\alpha = 0$, the maximum value of W_s is 1.000 for the M/M/1 model, while it is only 0.251 for the M/M/2 model. This demonstrates that the largest possible waiting time is significantly decreased when the two-server model is used. The maximum waiting time of the M/M/2 model is much smaller than the other models, so the performance of the M/M/2 model is better even if its minimum value is slightly higher for lower values of α-cut. At $\alpha = 1$, the expected value of W_s is 0.250 for M/M/1 and 0.148 for M/M/2. This is a demonstration of the two-server model's ability to shorten the waiting time of customers in the system.

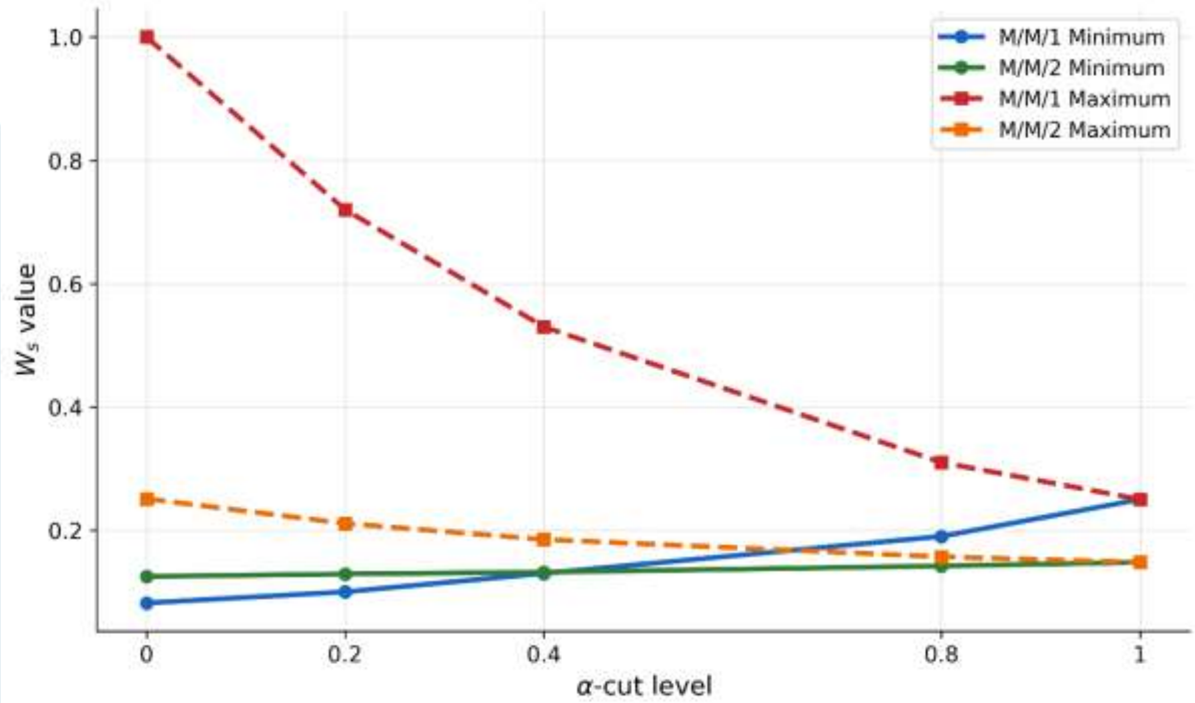

Figure 11. Comparison of Minimum and Maximum Values of $W_s$for Fuzzy $M/M/1$and Fuzzy $M/M/2$Models

This figure illustrates the waiting time in the system in the both fuzzy M/M/1 and fuzzy M/M/2 model. It is evident from the figure that the maximum waiting time in the M/M/2 model is considerably less. This indicates that with two servers, the time spent in the system will be reduced. Customers in the M/M/1 model may be required to wait a longer time as only one server is processing all customers. This makes the overall system time greater. With the M/M/2 model the work is divided between two servers, thus reducing the delay and also increasing the service speed. So, the figure indicates that the overall waiting time of customers can be reduced by the fuzzy M/M/2 model.

Table 6. Comparison of Minimum and Maximum Values of W_q

α	$M/M/1$**Minimum** W_q	$M/M/2$**Minimum** W_q	$M/M/1$**Maximum** W_q	$M/M/2$**Maximum** W_q

0	0.0207	0.0001	0.665	0.085
0.2	0.0280	0.0007	0.440	0.050
0.4	0.0400	0.0011	0.301	0.029
0.8	0.0590	0.0037	0.149	0.010
1	0.1070	0.0060	0.107	0.006

We compare minimum and maximum values of W_q for for the fuzzy M/M/1 and fuzzy M/M/2 model in Table 6. The value of W_q represents the average time waiting in the queue before service is started. This table clearly displays that the minimum and maximum values of W_q are is lower in the fuzzy M/M/2 model. At $\alpha = 0$, the maximum value of W_q is 0.665 for the M/M/1 model, while it is only 0.085 for the M/M/2 model. At $\alpha = 1$, the expected value of Wq is 0.107 for M/M/1 and only 0.006 for M/M/2. This shows a very large reduction in queue waiting time. It's because the second server decreases the possibility for customers to wait for service. Hence, the fuzzy M/M/2 model works much better in the sense of minimizing waiting time in the queue.

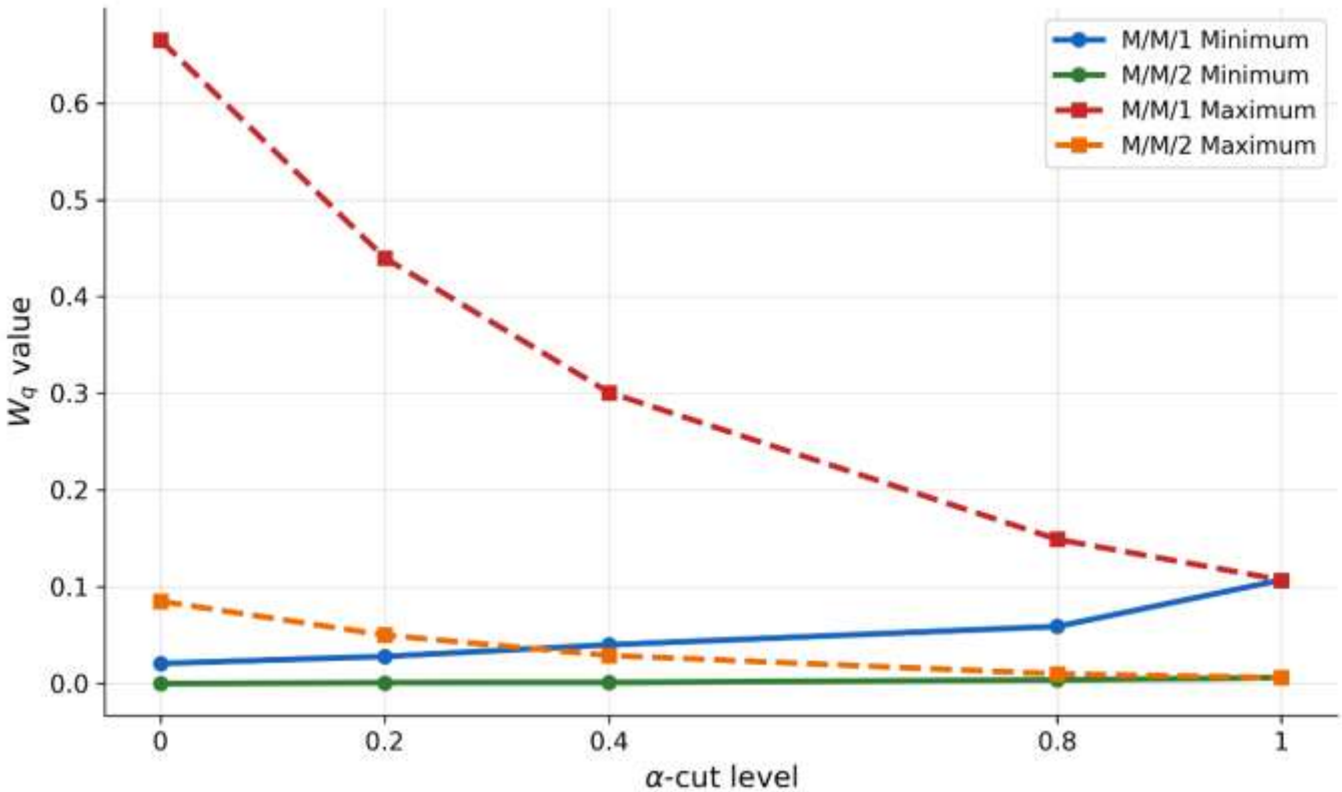

Figure 12. Comparison of Minimum and Maximum Values of $W_q$for Fuzzy $M/M/1$and Fuzzy $M/M/2$Models

This figure compares the waiting time in the queue for the fuzzy M/M/1 and fuzzy M/M/2 model. This figure illustrates that the results of the waiting times are lower for the M/M/2 model for both the minimum and maximum scenarios compared to the M/M/1 model. These results validate the improvement in the queueing performance as provided by the second server. The M/M/1 queue has a single server, which means that in this model customers must wait longer. If the server is full, all customers will have to queue up. Two servers are used in the M/M/2 model to decrease the wait time because the customers can receive services faster. Hence, this result indicates that further increase in the number of servers can considerably decrease queue delay.

The comparative analysis of the fuzzy M/M/1 and fuzzy M/M/2 queueing model reveals that in most performance measures, the fuzzy M/M/2 model out performs the fuzzy M/M/1 model. The two-server model gives lower values of L_q, L_s, W_s, and W_q, which means that it reduces queue length, system length, waiting time in the system, and waiting time in the queue. As seen from the results in the tables and figures, it is clear that the second server system enhances the capacity of the service in the system. The number of customers in the system is less, and the waiting time is less. The performance of the model M/M/2 is better and more trustworthy, although there are slight differences at lower α-cut values. Thus, it is concluded that the queueing system is better under fuzzy condition with two servers compared to one server and service efficiency is better.

13. "Discussion

The results show that fuzzy queueing models are useful for studying systems with uncertain arrival and service rates.[16] The α-cut method converts fuzzy values into interval values, making it possible to calculate lower and upper bounds for each performance measure. In the fuzzy $M/M/1$model, there is only one server.[17] Therefore, the queue length and waiting time are higher. In the fuzzy $M/M/2$model, two servers are available, so customers can be served faster. This reduces the number of customers waiting in the queue and also reduces the time spent in the system.[18] The comparison shows that the $M/M/2$model performs better than the $M/M/1$model in most performance measures.[19] The expected number of customers in the queue, expected number of customers in the system, waiting time in the queue, and waiting time in the system are all reduced when the number of servers increases.[20] This result is useful for real systems such as restaurants, billing counters, hospitals, toll plazas, and service centers.[21] If customer waiting time is high, adding another server can improve system performance. However, adding servers also increases operating cost.[22] Therefore, fuzzy queueing analysis can help decision makers balance service quality and cost."

14. Conclusion

[16] Kleinrock L. Queueing Systems, Volume I: Theory. New York: John Wiley & Sons; 1975.
[17] Little JDC. A proof for the queueing formula $L = \lambda W$. Operations Research. 1961;9(3):383-387.
[18] Kendall DG. Some problems in the theory of queues. Journal of the Royal Statistical Society Series B. 1951;13(2):151-185.
[19] Medhi J. Stochastic Models in Queueing Theory. 2nd ed. Amsterdam: Academic Press; 2003
[20] Taha HA. Operations Research: An Introduction. 10th ed. Boston: Pearson; 2017.
[21] Ross SM. Introduction to Probability Models. 11th ed. Amsterdam: Academic Press; 2014.
[22] Cox DR, Smith WL. Queues. London: Methuen; 1961.

This paper has investigated the fuzzy environment of M/M/1 and M/M/2 queuing models. Triangular fuzzy numbers were used to represent the arrival rate and the service rate, and the α-cut method was used to calculate interval values. The fuzzy performance measures were obtained by using the DSW algorithm. The numerical analysis shows that the fuzzy M/M/2model gives better performance than the fuzzy M/M/1model. The two server model is better in terms of the average number of customers in queue, average number of customers in system, average waiting time in the queue, and average waiting time in the system. The study proves that the fuzzy logic is useful in the applications of queuing system with uncertain arrival and service rates. It provides a more realistic picture of real service systems than one of crisp queueing models. The conclusion can assist the managers in determining whether they need to allocate more servers to minimize the waiting time and enhance the efficiency of the servers' services.

References

[1] Zadeh LA. Fuzzy sets. Information and Control. 1965;8(3):338-353.

[2] Zadeh LA. Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems. 1978;1(1):3-28.

[3] Buckley JJ. Elementary queueing theory based on possibility theory. Fuzzy Sets and Systems. 1990;37(1):43-52.

[4] Prade HM. An outline of fuzzy or possibilistic models for queueing systems. In: Wang PP, Chang SK, editors. Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems. New York: Plenum Press; 1980. p.147-153.

[5] Li RJ, Lee ES. Analysis of fuzzy queues. Computers & Mathematics with Applications. 1989;17(7):1143-1147.

[6] Negi DS, Lee ES. Analysis and simulation of fuzzy queues. Fuzzy Sets and Systems. 1992;46(3):321-330.

[7] Kao C, Li CC, Chen SP. Parametric programming to the analysis of fuzzy queues. Fuzzy Sets and Systems. 1999;107(1):93-100.

[8] Chen SP. Parametric nonlinear programming approach to fuzzy queues with bulk service. European Journal of Operational Research. 2005;163(2):434-444.

[9] Dubois D, Prade H. Fuzzy Sets and Systems: Theory and Applications. New York: Academic Press; 1980.

[10] Kaufmann A. Introduction to the Theory of Fuzzy Subsets. New York: Academic Press; 1975.

[11] Kaufmann A, Gupta MM. Introduction to Fuzzy Arithmetic: Theory and Applications. New York: Van Nostrand Reinhold; 1985.

[12] Zimmermann HJ. Fuzzy Set Theory and Its Applications. Dordrecht: Kluwer Academic Publishers; 1991.

[13] Dong WM, Shah HC, Wong FS. Fuzzy computations in risk and decision analysis. Civil Engineering Systems. 1985;2(4):201-208.

[14] Buckley JJ, Feuring T, Hayashi Y. Fuzzy queueing theory revisited. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 2001;9(5):527-537.

[15] Gross D, Shortle JF, Thompson JM, Harris CM. Fundamentals of Queueing Theory. 4th ed. Hoboken: John Wiley & Sons; 2008.

[16] Kleinrock L. Queueing Systems, Volume I: Theory. New York: John Wiley & Sons; 1975.

[17] Little JDC. A proof for the queueing formula $L = \lambda W$. Operations Research. 1961;9(3):383-387.

[18] Kendall DG. Some problems in the theory of queues. Journal of the Royal Statistical Society Series B. 1951;13(2):151-185.

[19] Medhi J. Stochastic Models in Queueing Theory. 2nd ed. Amsterdam: Academic Press; 2003

[20] Taha HA. Operations Research: An Introduction. 10th ed. Boston: Pearson; 2017.

[21] Ross SM. Introduction to Probability Models. 11th ed. Amsterdam: Academic Press; 2014.

[22] Cox DR, Smith WL. Queues. London: Methuen; 1961.